The CVXR Package

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What is "Convex Optimization" ?

"**Convex minimization** is a subfield of optimization that studies the problem of minimizing convex functions over convex sets."
–Wikipedia

- Very fast algorithms (like Linear Programming, LP)
- Convex problems have only *one* (global) optimum
- Many statistical and engineering applications can be modeled as convex problems
- BUT: May be difficult to find an appropriate convex formulation (NP-hard)

Convex Functions and Domains

A function $f : \mathbf{R}^n \to \mathbf{R}$ is *convex* if its domain of definition is convex and for all x, y and $0 \leq \theta \leq 1$ we have

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

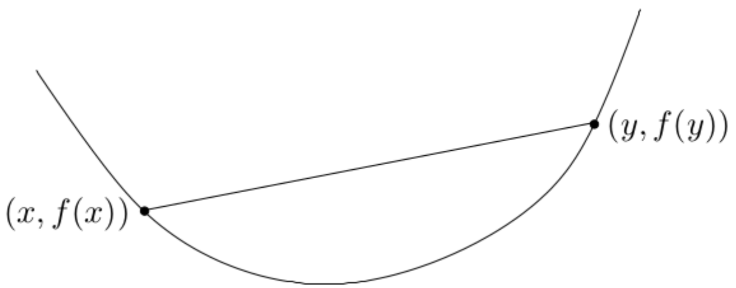


Figure 1: Figure: Graph of a convex function (Boyd et al. 2004)

What is CVX* ?

CVX* is a family of implementations of **Disciplined Convex Programming** (DCP), invented an initialized by *Stephen Boyd* and collaborators at Stanford University:

- CVX (MATLAB, ~2005)
- CVXPY (Python, 2013)
- convex.jl (Julia, 2015)
- CVXR (R, 2017)

Disciplined convex programming imposes a set of conventions to follow when constructing convex problems.

Loading CVXR

```
devtools::install_github("anqif/CVXR")
vignette("cvxr_intro", package="CVXR")
```

```
# suppressMessages(suppressWarnings(library(CVXR)))
library(CVXR)
```

```
##
## Attaching package: 'CVXR'

## The following object is masked from 'package:stats':
##
##     power
```

```
package?CVXR
```

Linear Regression with CVXR

```
x <- Variable(11)
objective  <- Minimize(sum((b - A %*% x)^2))
problem    <- Problem(objective)
result     <- solve(problem)
c( result$getValue(x) )
```

```
##  [1] -0.05059088 -1.95850906 -0.02934818  0.02498838 -0
##  [6]  0.00479077 -0.00087763  2.04205369  0.16839344  0
## [11]  0.36563333
```

Example: Linear Regression

```
wine <- read.csv("winequality.csv", sep=";")

mod0 <- lm(quality ~ . - 1, data=wine)
unname(coefficients(mod0))
```

```
##  [1] -0.05059062 -1.95851023 -0.02934924  0.02498840 -0.
##  [6]  0.00479079 -0.00087763  2.04204607  0.16839514  0.
## [11]  0.36563338
```

```
A <- wine[, 1:11]; b = wine[, 12]
mod00 <- qr.solve(A, b)
unname(mod00)
```

```
##  [1] -0.05059062 -1.95851023 -0.02934924  0.02498840 -0.
##  [6]  0.00479079 -0.00087763  2.04204607  0.16839514  0.
## [11]  0.36563338
```

Positive Coefficients only

```
x <- Variable(11)
objective   <- Minimize(sum((b - A %*% x)^2))
constraint <- list(x >= 0)
problem     <- Problem(objective, constraint)
result      <- solve(problem)
c( result$getValue(x) )
```

```
##  [1] -1.0927e-10  4.6149e-11  1.1556e-01  1.9532e-02  4.
##  [6]  5.0505e-03 -3.0693e-10  4.4630e-01  3.3277e-01  3.
## [11]  3.6525e-01
```

A 'Sum Equal to 1' Solution

```
x <- Variable(11)
objective  <- Minimize(sum((b - A %*% x)^2))
constraint <- list(x >= 0, sum(x) == 1)
problem    <- Problem(objective, constraint)
result     <- solve(problem)
zapsmall( c( result$getValue(x) ) )
```

```
##  [1] 0.00000 0.00000 0.00000 0.02209 0.00000 0.00554 0.(
##  [9] 0.46537 0.12725 0.37975
```

```
sum(result$getValue(x))
```

```
## [1] 1
```

L1 Regression

"L1 regression, or Least Absolute Deviations (LAD) regression, is a statistical optimality criterion and the statistical optimization technique that relies on minimizing the L1-norm."

Linear L1 regression: $\text{Min! } \sum_1^n |\, b - A\,x|$

```
x <- Variable(11)
objective  <- Minimize(sum(abs(b - A %*% x)))
constraint <- list(x[11] == 0)
problem    <- Problem(objective, constraint)
result     <- solve(problem)
c( result$getValue(x) )
```

```
##  [1] -1.0958e-02 -4.6122e-01  1.9429e-02 -2.0589e-03 -5
##  [6]  1.3345e-03 -7.1307e-04  6.2513e+00  5.5697e-02  1
## [11] -1.2074e-11
```

'Isotonic' Regression

"In statistics, isotonic regression or monotonic regression is the technique of fitting a free-form line to a sequence of observations under the [monotone] constraints." – Wikipedia

Example: `x[1]<=x[2]<=...<=x[n]`

```
x <- Variable(11)
objective  <- Minimize(sum((b - A %*% x)^2))
constraint <- list(diff(x) >= 0)
problem    <- Problem(objective, constraint)
result     <- solve(problem)
c( result$getValue(x) )
```

```
##  [1] -0.0212915 -0.0212915  0.0016911  0.0016911  0.001(
##  [7]  0.0016911  0.3767073  0.3767073  0.3767073  0.376
```

Robust Regression

"Robust regression is a form of regression analysis designed to overcome some limitations of traditional parametric and non-parametric methods, especially high sensitivity to outliers."

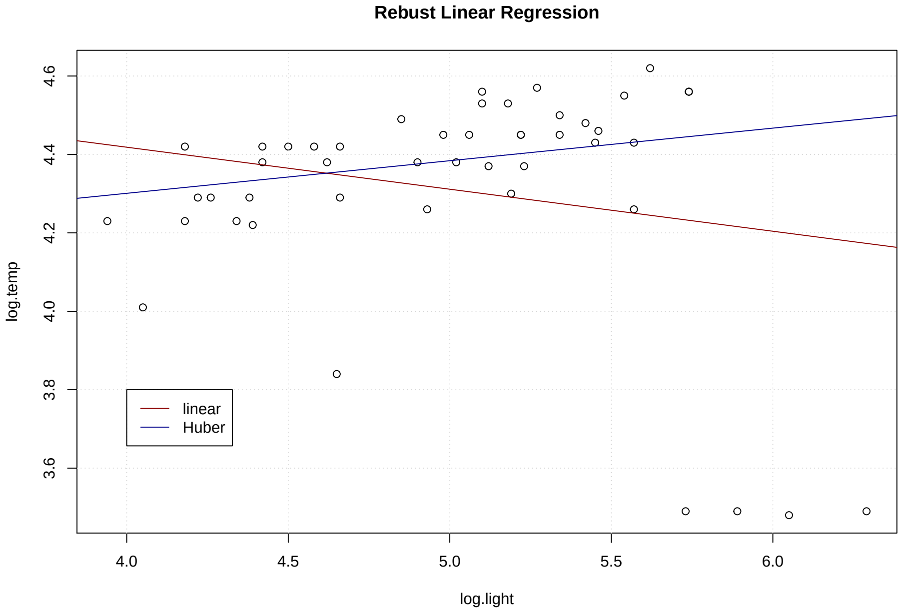
Huber's M-estimation: $\text{Min! } \sum L_M(b - A\,x)$ with $L_M(u) = \frac{1}{2}u^2$ if $|u| \leq M$, else $2M|u| - M^2$.

```
M <- 1   # Huber threshold
x <- Variable(11)
objective  <- Minimize(sum(huber(b - A %*% x, M)))
problem    <- Problem(objective)
result     <- solve(problem)
c( result$getValue(x) )
```

```
##  [1] -0.0478472 -1.8331217  0.0153206  0.0216320 -0.936(
##  [7] -0.0011800  1.8376328  0.2056466  0.4656358  0.366
```

Example: Robust Regression

Stars outer temperature vs. light intensity:



Example continued . . .

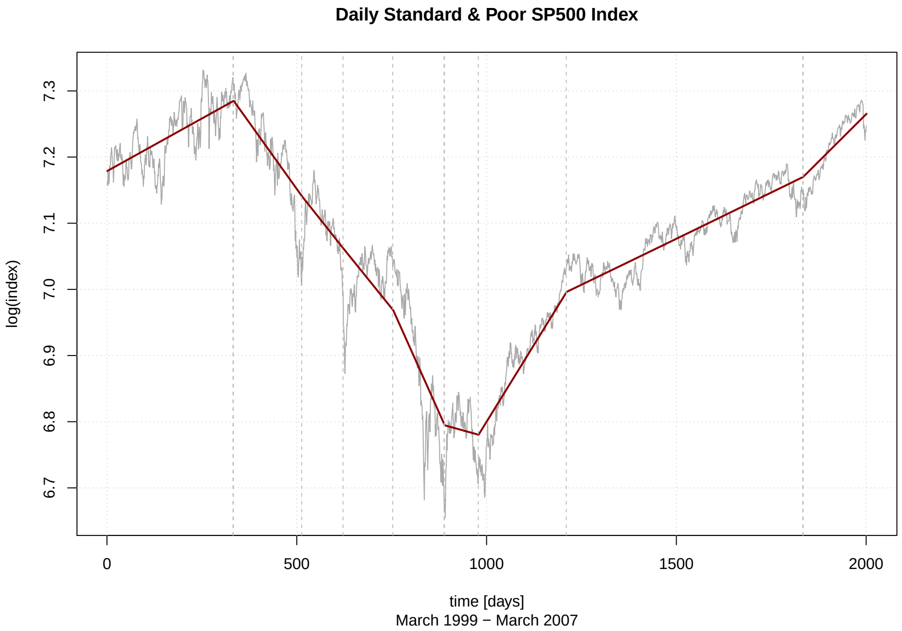
```
# library(CVXR)

stars = read.csv("starscyg.csv")
A = cbind(1, stars$log.light)
b = stars$log.temp

M <- 0.2  # Huber threshold
x <- Variable(2)
objective  <- sum(huber(b - A %*% x, M))
problem    <- Problem(Minimize(objective))
result     <- solve(problem)
ab = result$getValue(x)
ab
```

```
##           [,1]
## [1,] 3.968469
## [2,] 0.083105
```

Example: Piecewise Linear Regression



Example Solved with CVXR

One approach to ‘piecewise linear regression’ is through this formula:

$$\text{Min!}\frac{1}{2}\sum_{1}^{n}(y_i - z_i)^2 + \lambda \sum_{1}^{n-2} |z_i - 2z_{i+1} + z_{i+2}|$$

```
lambda = 40
z <- Variable(length(y))
objective <- 0.5 * p_norm(y - z) +
                lambda * p_norm(diff(z, differences = 2), 1
problem  <- Problem(Minimize(objective))
sol <- solve(problem)$getValue(z)
```

Quadratic Optimization

Quadratic Programming (QP) is the problem of optimizing a quadratic expression of several variables subject to linear constraints.

$$\text{Minimize} \quad \frac{1}{2}x^TQx + c^Tx \qquad s.t. \quad Ax \leq b$$

where
Q is a symmetric, positive (semi-)definite $n \times n$-matrix,
c an n-dim. vector,
A an $m \times n$-matrix, and
b an m-dim. vector.

For some solvers, linear equality constraints are also allowed.

Quadratic Optimization CRAN Optimization Task

Example (continued)

As an example, we will look at finding a smallest circle enclosing 100 randomly given points $p_1, \ldots, p_{100}$ in $\mathbb{R}^2$. We will represent the coordinates of these points as columns in the following matrix `P`.

```
set.seed(7531); N <-  100
P <- matrix(10*rnorm(2*N), nrow=2)
# plot(P[1, ], P[2, ], col="red", xlab="", ylab="")
```

```
C <- t(P) %*% P
d <- apply(P^2, 2, sum)
```

Example: Smallest Enclosing Ball

Given a set $P = \{p_1, ..., p_n\}$ of n points in $\mathbb{R}^d$, find a point p_0 such that $max||p_i - p_0||$ is minimized.

Known algorithm to solve this as Qudratic Programming task:

Define matrix $C = (p_1, ..., p_n)$, i.e. coordinates of points in columns, and minimize the quadratic form

$$x^TC^TCx - \sum p_i^Tp_ix_i$$

subject to $\sum x_1 = 1$ and all $x_i >= 0$.

Let $x = (x_1, ..., x_n)$ be an optimal solution, then the linear combination $p0 = \sum x_ip_i$ is the center of the smallest enclosing ball, and the negative of the minimum value at x is the square of the radius of the ball.

Example Solved with CVXR

```
x            <- Variable(N)
objective    <- Minimize( quad_form(x, C) - sum(d * x))
constraint   <- list(x >= 0, sum(x) == 1)
problem      <- Problem(objective, constraint)
result       <- solve(problem, solver="SCS")      # default:
```

```
x0 <- result$getValue(x)
p0 <- P %*% x0; c(p0)
```

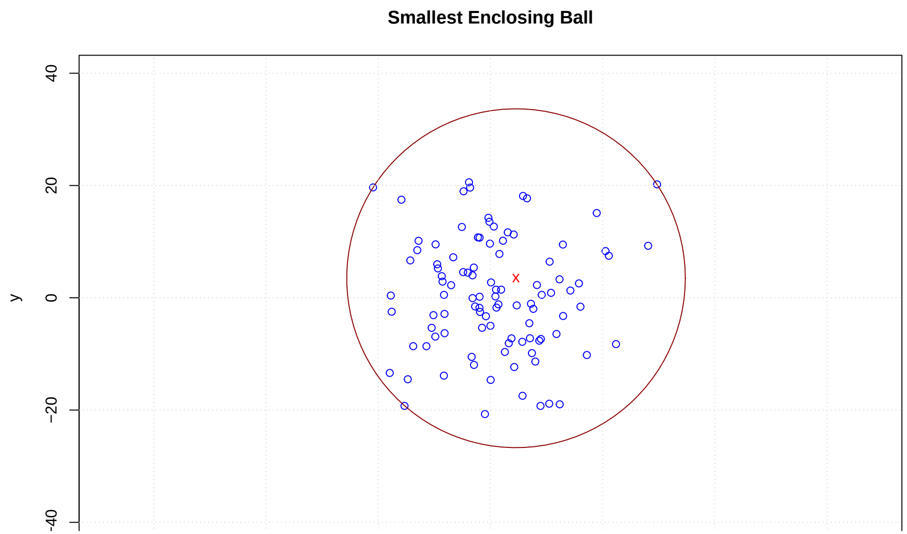
```
## [1] 4.5508 3.4853
```

```
r0 <- c(sqrt(sum(colSums(P^2)*x0) - t(x0)%*%t(P)%*%P%*%x0))
r0
```

```
## [1] 30.182
```

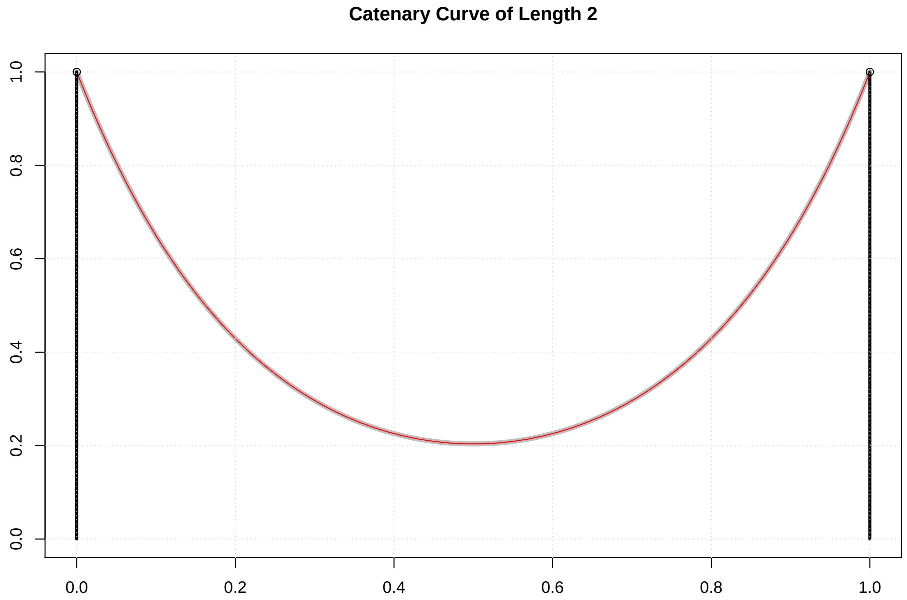
Example Solution

```r
plot(P[1, ], P[2, ], xlim=c(-40,40), ylim=c(-40,40), asp=1,
     col = "blue", xlab = "x", ylab = "y",
     main = "Smallest Enclosing Ball")
# ...
```



Example: Catenary

Solve the "hanging chain curve" as an optimization problem!
See hwborchers.lima-city.de/Presents/catenary.html.



CVXR Tutorial Examples

- Largest Euclidean ball in a 2D polyhedron
- Catenary Problem
- Huber Regression
- Logistic Regression
- Quantile Regression
- Censored Regression
- Isotonic Regression
- Near Isotonic and Near Convex Regression
- L1 Trend Filtering
- Elastic Net
- Saturating Hinges
- Direct Standardization
- Log-Concave Density Estimation
- Sparse Inverse Covariance Estimation
- Kelly Gambling
- Fastest Mixing Markov Chain
- Portfolio Optimization

Catenary Solved with CVXR

```r
N <- 100; L <- 2
h <- L / (N-1)
x <- Variable(N)
y <- Variable(N)
objective  <- Minimize(sum(y))
constraint <- list(x[1]==0, x[N]==1, y[1]==1, y[N]==1,
                   diff(x)^2 + diff(y)^2 <= h^2)
problem <- Problem(objective, constraint)
result <- solve(problem)    ## solver="SCS"
xm <- result$getValue(x)
ym <- result$getValue(y)
#  result
## $status:     "optimal"
## $solver:     "ECOS"
## $solve_time: 0.008145835
## $setup_time: 0.000476103
```

Running Times – Catenary Example

Solver	N = 50	N = 100	N = 1000
auglag	8.0	60 [–]	–
Ipopt			
CVXR/ECOS	0.283	0.297	NA
CVXR/SCS	0.311	0.330 [-]	1.141 [–]
ECOS	0.002	0.003	0.036
SCS	0.002	0.010	0.280
Rmosek	0.004	0.005	0.033
JuMP	0.007	0.016	0.416

Web Links

- See anqif/CVXR on Github
- CVXR Package vignette
- CVXR Home page
- CVXR Tutorial examples
- CVXR Function reference
- Anqi Fu's talk Disciplined Convex Optimization with CVXR at UseR!2016, Stanford University
- A. Fu, , B. Narasimhan, and Stephen Boyd. CVXR: An R Package for Disciplined Convex Optimization, Manuscript Draft, 2018.

Reference

A. Fu, B. Narasimhan, and S. Boyd (2018).*CVXR: An R Package for disciplined convex optimization*. Journal of Statistical Software. [To be published.]

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